Forest Chemicals Review
www.forestchemicalsreview.com
ISSN: 1520-0191
July-August 2022 Page No. 1802-1809
Article History: Received: 02 April 2022, Revised: 15 April 2022, Accepted: 24 April 2022, Publication: 04 May 2022

Oscillation Behavior for a Class of Second Order Neutral Differential Equation

Hongying Luo[1*], Jun Liu[2], Yuanhong Yu[3]
[1]College of Mathematics and Statistics, Qujing Normal University, Qujing, Yunna, China
[2]Education and Teaching Quality Control Center, Qujing Normal University, Qujing, Yunnan, China
[3]Academy of Mathematics and systems Science, China Academy Science, Beijing, China
*Corresponding Author.

Abstract:

Due to the extensive application of differential equation, the study of its vibration has been paid much attention. In this paper, we research the oscillation behavior of a class of second order dynamic models. This research is suit to the fields of engineering technique, aviation technology, automation control and so on. We analyse a class of second order neutral differential equation. First, we give a neutral dynamic model and analysis the research status of model. After that, we research its oscillation behavior and found a some of new sufficient conditions of the oscillatory property, a number of new Oscillation criteria for both $\beta \ge \alpha$ and $\beta < \alpha$ are founded for equation by making use of the generalized Riccati transformation and specific analytical skills. Our results are new and these results enrich and refine existing results. The results are verified by numerical experiments. Because of the neutral dynamic system model has an important status and action in engineering technique, aviation technology, automation control. As a result, our research is of big help to engineering technology, aviation technology, and automation control.

Keywords: *Neutral differential equation, Oscillation behavior, Generalized Riccati transformation.*

I. INTRODUCTION

In this paper, we research a class of second order neutral differential equation model:

$$\left(r(t)\left|z'(t)\right|^{\alpha-1} z'(t)\right)' + \int_c^d q(\xi, t)x^{\beta}(\sigma(\xi,t))d\xi = 0. \quad t_0 \le t. \tag{1}$$

Where $z(t) = x(t) + \int_a^b p(\xi,t)x(\tau(\xi,t))d\xi$, $0 \le a < b$, $0 \le c < d$, α and β are constants, and the following are satisfied:

(H_1) $p(\xi,t) \in C([a,b]\times[t_0,\infty),[0,\infty))$, $0 \le \int_a^b p(\xi,t)d\xi = p_0 < 1$, $q(\xi,t) \in C([c,d]\times[t_0,\infty),[0,\infty))$.

(H_2) $\tau(\xi,t) \in C([a,b]\times[t_0,\infty),(0,\infty))$, $\tau(\xi,t) \le t$, $\lim_{t\to\infty} \tau(\xi,t) = \infty$.

Forest Chemicals Review
www.forestchemicalsreview.com
ISSN: 1520-0191
July-August 2022 Page No. 1802-1809
Article History: Received: 02 April 2022, Revised: 15 April 2022, Accepted: 24 April 2022, Publication: 04 May 2022

(H_3) $r(t)\in C^1([t_0,\infty),(0,\infty))$, $r'(t)\ge 0$, $R(t)=\int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)ds$.

(H_4) $\sigma(\xi,t)\in C^1([c,d]\times[t_0,\infty),(0,\infty))$, $0<\sigma(\xi,t)\le t$, $\sigma_1(t)=\sigma(t,d)$, $\sigma_1'(t)>0$, $\lim_{t\to\infty}\sigma(t)=\infty$.

(H_5) $f:[t_0,\infty]\times R\to R$ is a continuous function such that $uf(u,t)>0$ for all $u\ne 0$ and there exists a function $q(\xi,t)\in C([c,d]\times[t_0,\infty])$ such that $|f(u,t)|\ge q(\xi,t)|u^{\beta}|$.

We refer to some relevant references, many references only study $\beta\ge\alpha$ or only study $\beta<\alpha$ this situation [1-4]. In this paper, we have not only studied $\beta\ge\alpha$, and also studied $\beta<\alpha$, a number of new oscillation criteria are founded for equation (1) by making use of the generalized Riccati transformation and specific analytical skills.

II.MAIN RESULTS

Lemma 1 [5]. Assume that $\theta>0$, $A>0$, $B\in R$, then $Bu-Au^{\frac{\theta+1}{\theta}}\le\frac{\theta^{\theta}}{(\theta+1)^{\theta+1}}\frac{B^{\theta+1}}{A^{\theta}}$.

Theorem 1. Assume that $\pi(t)=\int_{t}^{\infty} r^{-\frac{1}{\alpha}}(s)ds$, There exists $\rho(t)\in C([t_0,\infty],[0,\infty])$, such that

$$\int_{t_0}^{\infty}[Q_1(t)\rho(t)-\frac{r(\theta(t))(\rho'(t))^{\lambda+1}}{(\lambda+1)^{\lambda+1}(M\rho(t)\sigma_1(t))^{\lambda}}]dt=\infty. \tag{2}$$

$$\int_{t_0}^{\infty}[Q_2(t)\pi^{\delta}(t)-\frac{\mu}{\pi(t)r^{\frac{1}{\alpha}}(t)}]dt=\infty. \tag{3}$$

Where

$$\lambda=\min\{\alpha,\beta\},\delta=\max\{\alpha,\beta\},\quad \mu \text{ is an constant,}$$

$$Q_1(t)=\int_c^d q(\xi,t)[1-p(\sigma(\xi,t))]^{\beta}d\xi,\quad Q_2(t)=\int_c^d q(\xi,t)[1-\overline{p}(\sigma(\xi,t))]^{\beta}d\xi,$$

$$\overline{p}(t)=p(t)\frac{\pi(\tau_1(t))}{\pi(t)},\quad \theta(t)=\begin{cases}\sigma_1(t), & \beta\ge\alpha\\ t, & \beta<\alpha\end{cases},$$

Forest Chemicals Review
www.forestchemicalsreview.com
ISSN: 1520-0191
July-August 2022 Page No. 1802-1809
Article History: Received: 02 April 2022, Revised: 15 April 2022, Accepted: 24 April 2022, Publication: 04 May 2022

Then equation (1) is oscillatory.

Proof. Let $x(t)$ is an positive solution of equation (1) and $x(t) \geq 1$, there exists $t_1 \geq t_0$, such that, for $t \geq t_1$, we have

$$\left(r(t)\left|z'(t)\right|^{\alpha-1} z'(t)\right)' \leq 0.$$

Hence $r(t)\left|z'(t)\right|^{\alpha-1} z'(t)$ is decreasing function and it is fixed symbol function. Thus, there have Two possible situations for $z'(t)$, $z'(t) > 0$ or $z'(t) < 0$.

(i) Assume that $z'(t) > 0$ for $t_1 \leq t$, there have

$$z(t) = x(t) + \int_a^b p(\xi,t)x(\tau(\xi,t))d\xi \leq x(t) + \int_a^b p(\xi,t)z(\tau(\xi,t))d\xi$$
$$\leq x(t) + z(t)\int_a^b p(\xi,t)d\xi \leq x(t) + z(t)p(t).$$

Thus, from the last inequality, we have

$$x(t) \geq [1 - p(t)]z(t).$$

We obtain

$$\left(r(t)(z'(t))^{\alpha}\right)' + \int_c^d q(\xi,t)[1 - p(\sigma(\xi,t))]^{\beta} z^{\beta}(\sigma(\xi,t))d\xi \leq 0.$$

Or

$$\left(r(t)(z'(t))^{\alpha}\right)' + Q_1(t)z^{\beta}(\sigma_1(t)) \leq 0. \tag{4}$$

Let the function

$$w(t) = \rho(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(\sigma_1(t))}.$$

We have

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)}w(t) - Q_1(t)\rho(t) - \frac{\rho(t)r(t)(z'(t))^{\alpha}\beta z'(\sigma_1(t))\sigma_1'(t)}{z^{\beta+1}(\sigma_1(t))}.$$

$$\leq \frac{\rho'(t)}{\rho(t)}w(t) - Q_1(t)\rho(t) - \frac{\beta\sigma_1'(t)z'(\sigma_1(t))}{(\rho(t)r(t))^{1/\alpha}z'(t)}[z(\sigma_1(t))]^{\frac{\beta}{\alpha}-1}w^{1+\frac{1}{\alpha}}(t).$$

Case1: Assume that $\beta \geq \alpha$, thus there exists $M_1 > 0$, such that

Forest Chemicals Review
www.forestchemicalsreview.com
ISSN: 1520-0191
July-August 2022 Page No. 1802-1809
Article History: Received: 02 April 2022, Revised: 15 April 2022, Accepted: 24 April 2022, Publication: 04 May 2022

$$[z(\sigma_1(t))]^{\frac{\beta}{\alpha}-1} \geq M_1.$$

That is

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - Q_1(t)\rho(t) - \frac{M_1 \beta \sigma_1'(t) z'(\sigma_1(t))}{(\rho(t) r(t))^{1/\alpha} z'(t)} w^{1+\frac{1}{\alpha}}(t)$$
$$\leq \frac{\rho'(t)}{\rho(t)} w(t) - Q_1(t)\rho(t) - \frac{M_1 \alpha \sigma_1'(t) z'(\sigma_1(t))}{(\rho(t) r(t))^{1/\alpha} z'(t)} w^{1+\frac{1}{\alpha}}(t).$$

Since $r(t)(z'(t))^{\alpha}$ is decreasing function.

Thus, we have

$$r^{\frac{1}{\alpha}}(t) z'(t) \leq r^{\frac{1}{\alpha}}(\sigma_1(t)) z'(\sigma_1(t)), \quad \left(\frac{r(t)}{r(\sigma_1(t))}\right)^{\frac{1}{\alpha}} \leq \frac{z'(\sigma_1(t))}{z'(t)}.$$

Or

$$w'(t) \leq -Q_1(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha M_1 \sigma_1'(t)}{(\rho(t) r(\sigma_1(t)))^{1/\alpha}} w^{1+\frac{1}{\alpha}}(t). \tag{5}$$

Case 2: Assume that $\beta < \alpha$, we have

$$w'(t) \leq -Q_1(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\beta \sigma_1'(t) z'(\sigma_1(t))}{(\rho(t) r(t))^{1/\beta}} [z'(t)]^{1-\frac{\alpha}{\beta}} w^{1+\frac{1}{\beta}}(t).$$

Since $\left(r(t)(z'(t)^{\alpha}\right)' \leq 0$, we have

$$r'(t)(z'(t))^{\alpha} + \alpha(z'(t))^{\alpha-1} z''(t) r(t) \leq 0.$$

Hence, we get $z''(t) \leq 0$, $z'(t)$ is decreasing function, thus that, $z'(t) \leq z'(\sigma_1(t))$, we have

$$w'(t) \leq -Q_1(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\beta \sigma_1'(t)}{(\rho(t) r(t))^{1/\beta}} [z'(t)]^{1-\frac{\alpha}{\beta}} w^{1+\frac{1}{\beta}}(t).$$

Since $z'(t) > 0$, and $z'(t)$ is decreasing function, hence $[z(t)]^{1-\frac{\alpha}{\beta}}$ is increasing function, thus there exists a constant $M_2 > 0$, such that $[z(t)]^{1-\frac{\alpha}{\beta}} \geq M_2$, we have

$$w'(t) \leq -Q_1(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\beta M_2 \sigma_1'(t)}{(\rho(t) r(t))^{1/\beta}} w^{1+\frac{1}{\beta}}(t). \tag{6}$$

Forest Chemicals Review
www.forestchemicalsreview.com
ISSN: 1520-0191
July-August 2022 Page No. 1802-1809
Article History: Received: 02 April 2022, Revised: 15 April 2022, Accepted: 24 April 2022, Publication: 04 May 2022

From (5) and (6), let $\lambda = \min\{\alpha, \beta\}$, $M = \min\{\alpha, \beta\}$, $\theta(t) = \begin{cases} \sigma_1(t), & \beta \ge \alpha \\ t, & \beta < \alpha \end{cases}$, we have

$$w'(t) \le -Q_1(t)\rho(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\lambda M \sigma_1'(t)}{(\rho(t) r(\theta(t)))^{1/\lambda}} w^{\frac{\lambda+1}{\lambda}}(t). \tag{7}$$

Using lemma 1, thus, we have

$$w'(t) \le -Q_1(t)\rho(t) + \frac{r(\theta(t))(\rho'(t))^{\lambda+1}}{(\lambda+1)^{\lambda+1}(M\rho(t)\sigma_1'(t))^{\lambda}}. \tag{8}$$

Integrting (8) from T to t and using (2), we obtain

$$w(t) \le w(T) - \int_T^t [\rho(s)Q_1(s) - \frac{r(\theta(s))(\rho'(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}(M\rho(s)\sigma_1'(s))^{\lambda}}]ds \to -\infty \text{ as } t \to \infty.$$

Which contradicts to the fact $w(t) > 0$, therefore, $z'(t) > 0$ is impossible.

(ii) Assume that $z'(t) < 0$ for $t_1 \le t$, we have

$$\left(r(t)(-z'(t))^{\alpha}\right)' \ge 0$$

Thus $r(t)(-z'(t))^{\alpha}$ is an increasing function, for $s \ge t \ge T$, we have

$$r^{\frac{1}{\alpha}}(s)(-z'(s)) \ge r^{\frac{1}{\alpha}}(t)(-z'(t))$$

Or

$$z'(s) \le r^{-\frac{1}{\alpha}}(s) r^{\frac{1}{\alpha}}(t) z'(t)$$

We obtain

$$z(l) - z(t) \le r^{\frac{1}{\alpha}}(t) z'(t) \int_t^l r^{-\frac{1}{\alpha}}(s)ds$$

Or

$$z(t) \ge r^{\frac{1}{\alpha}}(t)(-z'(t))\pi(t). \tag{9}$$

Where

$$\pi(t) = \int_t^{\infty} r^{-\frac{1}{\alpha}}(s)ds.$$

Forest Chemicals Review
www.forestchemicalsreview.com
ISSN: 1520-0191
July-August 2022 Page No. 1802-1809
Article History: Received: 02 April 2022, Revised: 15 April 2022, Accepted: 24 April 2022, Publication: 04 May 2022

Since

$$z(t) \le x(t) + p(t)\frac{\pi(\tau_1(t))}{\pi(t)} z(t) \le x(t) + \overline{p}(t) z(t).$$

Or

$$x(t) \ge (1-\overline{p}(t))z(t), \qquad x^{\beta}(\sigma(\xi,t)) \ge (1-\overline{p}(\sigma(\xi,t))^{\beta} z^{\beta}(\sigma(\xi,t).$$

We have

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}(-z'(t))\right)' - Q_2(t) z^{\beta}(\sigma_1(t)) \ge 0.$$

Where

$$Q_2(t) = \int_c^d q(\xi,t)[1-\overline{p}(\sigma(\xi,t))]^{\beta} d\xi, \quad \sigma_1(t) = \sigma(t,d)$$

Since $\sigma_1(t) \le t$, $z'(t) < 0$, thus that $z(\sigma_1(t)) \ge z(t)$, we have

$$\left(r(t)(-z'(t))^{\alpha}\right)' \ge Q_2(t) z^{\beta}(t)$$

We using the generalized Riccati transformation

$$V(t) = \frac{r(t)(-z'(t))^{\alpha}}{z^{\beta}(t)}.$$

Case1: Assume that $\beta < \alpha$, we have

$$V'(t) \ge Q_2(t) + \frac{\beta r(t)(-z'(t))^{\alpha+1}}{z^{\beta+1}(t)} \ge Q_2(t) + \frac{M_1}{r^{\frac{1}{\alpha}}(t)} V^{\frac{\alpha+1}{\alpha}}(t) \tag{10}$$

Where M_1 is a constant.

Case2: Assume that $\beta \ge \alpha$, we have

$$V'(t) \ge Q_2(t) + \frac{\beta(-z'(t))^{1-\frac{\alpha}{\beta}}}{r^{\frac{1}{\beta}}(t)} V^{\frac{\beta+1}{\beta}}(t) \ge Q_2(t) + \frac{M_2}{r^{\frac{1}{\alpha}}(t)} V^{\frac{\beta+1}{\beta}}(t). \tag{11}$$

Where M_1 is a constant.

Let $M = \min\{M_1, M_2\}$, $\delta = \max\{\alpha, \beta\}$, from (10) and (11), we have

Forest Chemicals Review
www.forestchemicalsreview.com
ISSN: 1520-0191
July-August 2022 Page No. 1802-1809
Article History: Received: 02 April 2022, Revised: 15 April 2022, Accepted: 24 April 2022, Publication: 04 May 2022

$$V'(t) \geq Q_2(t) + \frac{M}{r^{\frac{1}{\alpha}}(t)} V^{\frac{\delta+1}{\delta}}(t). \tag{12}$$

From (9), when $\alpha > \beta$, we have

$$L_1 \geq z^{\alpha-\beta}(t) \geq \pi^{\alpha}(t)V(t))$$

When $\alpha \leq \beta$, we have

$$L_2 \geq \frac{1}{r^{\frac{\beta}{\alpha}-1}(-z'(t))} \geq \pi^{\beta}(t)V(t))$$

Where L_1, L_2 are constants, let $L = \max\{L_1, L_2\}$, we have

$$\pi^{\delta}(t)V(t)) \leq L$$

From (12), we have

$$\pi^{\delta}(t)Q_2(t) \leq \pi^{\delta}(t)V'(t) - \frac{M\pi^{\delta}(t)}{r^{\frac{1}{\alpha}}(t)} V^{\frac{\delta+1}{\delta}}(t)$$

Or

$$\begin{aligned} \int_T^t \pi^{\delta}(s)Q_2(s)ds &\leq \int_T^t \pi^{\delta}(s)V'(s)ds - \int_T^t \frac{M\pi^{\delta}(s)}{V^{\frac{1}{\alpha}}(s)} V^{\frac{\delta+1}{\delta}}(s)ds \\ &\leq 2M + \int_T^t \pi^{\delta-1}(s)r^{-\frac{1}{\alpha}}(s)[\delta V(s) - M\pi(s)V^{\frac{\delta+1}{\delta}}(s)]ds \\ &\leq 2M + \int_T^t \frac{\mu}{\pi(s)r^{\frac{1}{\alpha}}(s)} ds \end{aligned}$$

Where $\mu = (\frac{\delta}{\delta+1})^{\delta+1}(\frac{\delta}{M})^{\delta}$

Therefore, we have

$$\int_T^t [\pi^{\delta}(s)Q_2(s) - \frac{\mu}{\pi(s)r^{\frac{1}{\alpha}}(s)}]ds \leq 2M$$

The above inequality is contradictory to condition (3), therefore, the proof is complete.

ISSN: 1520-0191
July-August 2022 Page No. 1802-1809
Article History: Received: 02 April 2022, Revised: 15 April 2022, Accepted: 24 April 2022, Publication: 04 May 2022

III. CONCLUSION

In this paper, We have not only studied $\beta \geq \alpha$ and also studied $\beta < \alpha$, a number of new oscillation criteria are founded for equation (1) by making use of the generalized Riccati transformation and specific analytical skills.

ACKNOWLEDGEMENTS

This research was supported by National Natural Science Foundation of China (Grant No. 11361048), Yunnan Natural Science Foundation (Grant No. 2017FH001-014, 2020CXTD25, 2022J0792, 2022J0805), and Qujing Normal University Science Foundation (Grant No.ZDKC2016002, JGXM201804, QJQSKT2021ZD002, QJQSKT2021YB02).

REFERENCES

[1] Candan T, Oscillatory behavior of second order nonlinear neutral differential equations with distributed deviating arguments. Applied Mathematics and Computation, 262 (2015), pp. 199-203.
[2] Dzurina J, Stavroulakis L P, Oscillatory criteria for second order delay differential equations. Applied Mathematics and Computation, 140 (2003), pp. 445-453.
[3] Cuimei J, Ying J, Tongxing L, Asymptotic behavior of third order differential equations with nonpositi ve neutral coefficients and distributed deviating arguments. Advances in Difference Equations, 105(2016), pp. 1-14.
[4] Li W T, Interval oscillation of second-order half-linear functional differential equations. Applied Mathematics and Computation, 155 (2004), pp. 451–468.
[5] Liu H, Meng F, Liu P, Oscillation and asymptotic analysis on a new generalized Emden–Fowler equation, Applied Mathematics and Computation 219 (2012), pp. 2739–2748.